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GROUPS AND MONOIDS AND THEIR ALGEBRAS
A COHOMOLOGICAL STUDY I

by

Olav Arnfinn Laudal

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Introduction

The following pages contain the three first paragraphs of a study on the cohomological properties of groups and monoids and their associated algebras.

The starting point of this study was the realization that the Massey-type product structure of the cohomology of a p-group determines quite a bit of the structure of the group, see [La 3].

This together with the fact that the classical Massey products only depend upon the group-algebra, made me hope that a finer study of the cohomology product structure would lead to a solution of the isomorphism problem for p-groups.

The following paper will show that I did not succede.

However, it seems to me that some of the results still merit publication, in particular since they have applications in other directions.

It turns out that the results of Chapter 1 may be, successfully, used in the study of the Bettiserics of two-dimensional torus imbeddings, and in the study of deformations of torus imbeddings, in general.

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Chapter 1. Cohomology of groups and algebras

The object of this chapter is firstly to prepare the ground for the next two chapters, secondly to prove the existence of a canonical isomorphism between the cohomology of an abelian monoid and the cohomology of the corresponding monoid-algebra, (1.3).
This isomorphism will be usefull in the study of deformations of monoid algebras, such as affine torus imbeddings.

We start by constructing a cohomology theory for groups, applying the method of model categories and the corresponding derived functors of the projective limit functor as in [An], [La 1], [La 2].

We then show the relationship between this kind of cohomology and the classical cohomology of groups, suggested by Barr and Reinhardt in [B-R].

In (1.2) we copy the procedure in (1.1) for the case of not necessarily commutative algebras, and in (1.3) we relate the cohomology of a group to the cohomology of the corresponding group-k-algebra. The main theorem applies to monoids as well.

Finally we add an appendix on the Betti-numbers of monoid-algebras.

(1.1) Cohomology of groups and monoids

If S is any set, we may consider the free group $F(S)$ and the free abelian group $F_{ab}(S)$ generated by S. The full subcatgories of gr, respectively abgr, generated by the free groups will be denoted by free respectively free ab.

Given a morphism of groups $P \overset{\delta}{\to} G$ we shall be interested in the category P-gr/G of all commutative diagrams in gr of the form

$$(1) \qquad f: \begin{tikzcd} P \arrow[rr, "\phi"] \arrow[dr, "\delta"'] & & Q \arrow[dl, "\delta'"] \\ & G & \end{tikzcd}$$

The full subcategory of P - gr/G generated by the diagrams (1) for which $Q = F(S)*P$ and ϕ is the canonical morphism of P into the direct sum $F(S)*P$, will be called P-free/G.

For the purpose of studying deformations, i.e. composite extensions, of groups, we need to consider a slightly more elaborate category.

Definition (1.1.1) A morphism of groups $\delta: P \to G$, together with a G-action η: Hom(G,Aut(P)), on P is called a normal morphism if for every $p \in P$ and every $g \in G$, $g\delta(p)g^{-1} = \delta(\eta(g)(p))$.

Given two normal morphisms $(\delta,\eta): P \to G$ and $(\delta',\eta'): P \to Q$ a morphism $\phi: (\delta',\eta') \to (\delta,\eta)$ is a commutative diagram

$$(2) \qquad f: \begin{tikzcd} & Q \arrow[d, "\phi"] \\ P \arrow[ur, "\delta'"] \arrow[r, "\delta"'] & G \end{tikzcd}$$

with $\eta' = \phi\circ\eta: G \to \mathrm{Aut}(P)$.

Denote by n-P - gr/G the category of such diagrams.

The diagrams (2) in which $Q \overset{\phi}{\to} G$ is the quotient of an object of P-free/G, $F(S)\ P \to G$, by the normal subgroup $\sim$ generated by the elements of the form $fpf^{-1}\eta(\vec{\phi}(f))(p)$ $f \in F(S)$ and $p \in P$, generate a full subcategory n-P-free/G of n-P-gr/G.

Now, let G-bimod be the category of G-bimodules. An object M of G-bimod is a k-module with commuting left and right G-actions.

Given a G-bimodule M and an object (1) respectively (2), we denote by

$$\mathrm{Der}_P(f,M) = \{D \in \mathrm{Set}(Q,M) \mid \text{ for all } q_1,q_2 \in Q$$
$$D(q_1 \cdot q_2) = \phi(q_1)D(q_2) + D(q_1)\phi(q_2),\ \delta' \circ D = 0\},$$

the k-module of derivations on Q vanishing on P.

Assume P operates trivially on M via δ, then for any $p \in P$, $q \in Q$ and $D \in \mathrm{Der}_P(f,M)$ we find $D(q\delta'(p)q^{-1}) = 0$.

Therefore any derivation on $F(S)*P = Q'$, see above, vanishing on P will factorize through $Q = F(S)*P/\sim$.

Obviously $\mathrm{Der}_P(G,M)$ is a k-sub-module of the k-module $\mathrm{Sets}(G,M) \simeq M^{|G|}$. Put $M^G = \{m \in M \epsilon g \in G,\ gm-mg = 0\}$ and observe that there is a natural k-linear homomorphism $\nu\colon M \to \mathrm{Der}_P(G,M)$ defined by $\nu(m)(g) = gm-mg$, $m \in M$, $g \in G$. Notice that if P is the trivial group, ν induces the exact sequence

$$(3) \qquad 0 \to M^G \to M \xrightarrow{\nu} \mathrm{Der}(G,M) \to H^1(G,M) \to 0$$

where $H^1(G,M)$ is the ordinary cohomology of G provided the right action of G on M is trivial.

In general, the correspondence $f \to \mathrm{Der}_P(f,M)$ defines a contravariant functor

P-free/G $\to$ k-mod

or

n-P-free/G $\to$ k-mod

depending on which category one chooses to consider.

The construction of the cohomology, and the elementary results we are going to prove in this paragraph, will be complete analogues whether we choose one or the other. In fact, we might also consider the commutative case without much change.

Therefore we shall focus on the case of a normal morphism $P \overset{\delta}{\to} G$, thus on the functor:

$$\mathrm{Der}_P(-,M)\colon (n\text{-}P\text{-}\underline{\mathrm{free}}/G)^{\circ} \to k\text{-}\underline{\mathrm{mod}}$$

<u>Definition</u> (1.1.2) Suppose $\delta\colon P \to G$ is a normal morphism and M a G-bi-module such that P, via δ, operates trivially on M. Then the algebra cohomology of δ with values in M is the graded k-module

$$A^{\bullet}(P,G;M) = \varprojlim_{(n\text{-}P\text{-}\underline{\mathrm{free}}/G)^{\circ}}{}^{(\bullet)} \mathrm{Der}_P(-,M).$$

When P is the trivial group, we write

$$A^{\bullet}(G;M) = A^{\bullet}(\{1\},G;M).$$

<u>Proposition</u> (1.1.3) "Leray spectral sequence". Let $\delta\colon F(S)*P/\sim \twoheadrightarrow G$ be a surjective normal P-morphism of groups. Put $F_o = F(S)*P/\sim$ and consider the semi-simplicial group

$$F.\colon G \leftarrow\!-\!- F_o \overset{\leftarrow}{\leftarrow} F_o \underset{G}{\times} F_o \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} F_o \underset{G}{\times} F_o \underset{G}{\times} F_o \cdots$$

Then there is a spectral sequence with

$$E_2^{p,q} = H^p(\overline{A^q(P,F.;M)})$$

converging to the cohomology $A^{\bullet}(P,G;M)$.

<u>Proof</u>. This is a trivial consequence of the Leray spectral sequence (2.1.3) of [La 2].

Q.E.D.

Proposition (1.1.4) With the hypothesis of (1.1.2) we have:

(i) Any short exact sequence of G-bimodules induces a long exact sequence in cohomology.

(ii) $A^o(P,G;M) = \mathrm{Der}_P(G,M)$

$$A^1(P,G;M) = \operatorname{Hom}_{F_o\text{class}}(J,M)/\mathrm{Der}_P$$

where J is the kernel of any surjective morphism of P-groups:

$$\phi_1 : F_o = F(S_1) * P/\sim \twoheadrightarrow G.$$

$$A^2(P,G;M) = \operatorname{Hom}_{F_1\text{-class}}(R_1/R_o, M)/\mathrm{Der}_P$$

where R_1 is the kernel of any surjective morphism of P-groups: $\phi_2 : F_1 = F(S_2) * P/\sim \to F_o \times J$.

Proof. (i) is trivial, as is the first assertion under (ii). The last two formulas follow from (1.1.3) by streightforward computation, see [La 2, (5.1)].

Proposition (1.1.5) Assume, in the situation of (1.1.3) that the right action of G on M is trivial. Then there are canonical isomorphisms

$$A^n(G,M) \simeq H^{n+1}(G,M) \quad \text{for } n > 1$$

Moreover, given an exact sequence of G-modules

$$0 \to M' \to M \to M'' \to 0$$

there is a commutative diagram of the form:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M'^G \arrow[r] \arrow[d] & M^G \arrow[r] \arrow[d] & M''^G \arrow[d] & & \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & A^o(G,M') \arrow[r] \arrow[d] & A^o(G,M) \arrow[r] \arrow[d] & A^o(G,M'') \arrow[r] \arrow[d] & A^1(G,M') \arrow[r] \arrow[d, equal] & \cdots \\
 & H^1(G,M') \arrow[r] \arrow[d] & H^1(G,M) \arrow[r] \arrow[d] & H^1(G,M'') \arrow[r] \arrow[d] & H^2(G,M') \arrow[r] & \cdots \\
 & 0 & 0 & 0 & &
\end{tikzcd}$$

in which all sequences are exact. The obvious snakes lemma produces the long exact sequence of ordinary cohomology.

Proof. Given an object of $\underline{gr}/G$, i.e. any morphism of groups $H \to G$ consider the non-homogenous complex $C^{\bullet}(H,M)$, see [C.-E. p.174]. We need to know that for $p \geq 1$ $C^p(H,M) = \text{Sets}(\underbrace{H\times\cdots\times H}_{p},M)$ and that

$$\text{Der}(H,M) = \ker(C^1(H,M) \overset{\delta}{\to} C^2(H,M))$$

$$H^i(H,M) = H^i(C^{\bullet}(H,M)) \quad \text{for } i \geq 1$$

Moreover, for H free, $H^i(H,M) = 0$ for all $i \geq 2$. Consequently $\text{Der}(H,M) \to C^{\bullet}(H,M)_{i\geq 1}$ is a resolution of $\text{Der}(H,M)$ for all M and all free groups H.

We obtain a functor

$$C^{\bullet}(-,M)\colon (\underline{\text{free}}/G)^{\circ} \to \underline{\text{compl}}.\ k\text{-}\underline{\text{mod}}$$

and we may consider the double complex

$$K^{\bullet\bullet} = C^{\bullet}((\underline{\text{free}}/G)^{\circ}, C^{\bullet}(-,M))$$

There are, as usual, two spectral sequences given by,

$$'E_2^{p,q} = \begin{cases} 0 & q \neq 1 \\ \varprojlim\limits_{(\underline{\text{free}}/G)^{\circ}}{}^{(p)} \text{Der}(-,M) = A^p(G,M) & q = 1 \end{cases}$$

$$''E_2^{p,q} = H^p(\varprojlim\limits_{(\underline{\text{free}}/G)^{\circ}}{}^{(q)} C^{\bullet}(-,M))$$

Both converging to the cohomology of the double complex $K^{\bullet\bullet}$.

The first part of the proposition is a consequence of the following,

Lemma (1.1.6) With the above assumptions, we have

(i) $$\varprojlim\limits_{(\underline{\text{free}}/G)^{\circ}} C^{\bullet}(-,M) = C^{\bullet}(G,M)$$

(ii) $$\varprojlim\limits_{(\underline{\text{free}}/G)^{\circ}}{}^{(q)} C^p(-,M) = 0 \quad \text{for } q \geq 1 \text{ and } p \geq 1.$$

Proof of lemma. For every $p \geq 1$, and every element $\underline{g} = (g_1,\ldots,g_p \in G^p$, let the morphism

$$E(\underline{g})\colon F(p) = F(\{1,2,\ldots,p\}) \to G$$

be defined by $E(\underline{g})(i) = g_i$.

Consider $E(\underline{g})$ as an object of $\underline{\text{free}}/G$, and let $\delta: F \to G$ be any other object, then

$$\mathrm{Mor}(E(\underline{g}),\delta) = \{(f_1,\dots,f_p) \in \underbrace{F\times\cdots\times F}_{p} \mid \delta(f_i) = g_i,\ i=1,\dots,p\}.$$

Therefore the union $\bigcup_{\underline{g}\in G^p} \mathrm{Mor}(E(\underline{g}),\delta')$ considered as a union of subsets of F^p is equal to F^p. Consequently, the Π-flabby object of the category $\underline{a} = k\text{-}\underline{\text{mod}}^{(\underline{\text{free}}/G)^o}$ of all functors $(\underline{\text{free}}/G)^o \to k\text{-}\underline{\text{mod}}$, defined by the objcts $E(g)$ and the k-module M see [La 1, p. 255], is given by

$$\delta \to \prod_{\bigcup_{\underline{g}\in G^p} \mathrm{Mor}(E(\underline{g}),\delta)} M = \prod_{F^p} M = C^p(F,M)$$

Thus $C^p(-,M)$ is Π-flabby, and it follows easily that

$$\varprojlim_{(\underline{\text{free}}/G)^o}{}^{(q)} C^p(-.M) = 0 \quad \text{for} \quad q > 1.$$

For $q = 0$ we find, $\varprojlim_{(\underline{\text{free}}/G)^o} C^{\bullet}(-,M) = \mathrm{Mor}_{\underline{a}}(k,C^p(-,M)) =$

$$= \prod_{G^p} \mathrm{Mor}_{k\text{-}\underline{\text{mod}}}(k,M) = C^p(G,M)$$

This proves the lemma. Q.E.D.

The last assertion of the proposition follows from the existence of the exact sequences (3), and easy diagram chasing. Q.E.D.

Interlude on monoids. Given a morphism $\Gamma \xrightarrow{\delta} \Lambda$ of monoids a Λ-bi--module M is a k-module with commuting left and right Λ-actions. We define in exactly the same way as above the categories $\Gamma\text{-}\underline{\text{free}}/\Lambda \subseteq \Gamma\text{-}\underline{\text{mon}}/\Lambda$ and the functor $\mathrm{Der}_{\Gamma}(-,M):(\Gamma\text{-}\underline{\text{free}}/\Lambda)^o \to k\text{-}\underline{\text{mod}}$. Therefore we may copy the definition of the cohomology groups,

$$A^{\bullet}(\Gamma,\Lambda;M) = \varprojlim_{(\Gamma\text{-}\underline{\text{free}}/\Lambda^o}{}^{(\bullet)} \mathrm{Der}_{\Gamma}(-,M)$$

and prove the same kind of results as in the case of groups. In particular the propositions (1.1.3) and (1.1.4) have obvious analogies. We shall not insist upon the details.

However, in the case of monoids, we shall be particularly interested in the commutative situation.

Suppose therefore that $\Gamma = \{1\}$ and Λ is a commutative monoid. Rephrasing (1.1.3) we find the following result:

There is a spectral sequence with

$$E_2^{p,q} = \overline{H^p(A^q(F\cdot;M))}$$

converging to $A^{\bullet}(\Lambda;M)$, where $F_o \twoheadrightarrow \Lambda$ is a surjective homomorphism of a free commutative monoid F_o onto Λ, and $F\cdot$ the simplicial monoid

$$\begin{array}{ccccccc} F_o & \leftleftarrows & F_o \times_\Lambda F_o & \overset{\leftarrow}{\leftleftarrows} & F_o \times F_o \times F_o & \cdots \\ & & \| & & \| & \\ & & F_1 & & F_2 & \cdots \end{array}$$

Example (1.1.7). If Λ is the submonoid of Z^n generated by $\underline{a}_1,\ldots,\underline{a}_r \in Z^n$, pick $F_o = Z_+^r$ with the map $\delta\colon F_o \twoheadrightarrow \Lambda$ defined by $\delta((0,\ldots,\underset{i}{1},\ldots.0)) = \underline{a}_i$. Then $F_o \underset{\Lambda}{\times} F_o = \{((f_1,\ldots,f_r),(g_1,\ldots,g_2)) \in Z_+^r \times Z_+^r \mid \sum_{i=1}^{r} f_i\underline{a}_i = \sum_{i=1}^{r} g_i\underline{a}_i\}$. There is a map $\eta\colon F_o \to Z^r$ defined by $(\underline{f},\underline{g}) \to (\underline{f}-\underline{g}) = (f_1-g_1,\ldots,f_r-g_r)$. The image is $J = \{(n_1,\ldots,n_r) \in Z^r \mid \sum_{i=1}^{r} n_i\underline{a}_i = 0\}$. Any element of $F_o \underset{\Lambda}{\times} F_o$ may be written as $(\underline{f},\underline{g}) \to (h,h) + (\underline{f}',\underline{g}')$ where $f_i',g_i' = 0$ for $i = 1,\ldots,r$. If $D \in \mathrm{Der}(F_o \times_\Lambda F_o, M)$ then $D((\underline{f},\underline{g})) = (\underline{h})\cdot D((\underline{f}',\underline{g}')) + D((\underline{h},\underline{h}))\cdot(\underline{f}')$. If, moreover, $D \in \ker(\mathrm{Der}(F_1,M) \to \mathrm{Der}(F_2,M))$ then one checks easily that $D((\underline{h},\underline{h})) = 0$ for all $\underline{h} \in F_n$. Therefore such a D is uniquely determined by its values on elements of the form $(\underline{f}',\underline{g}')$ with $f_i\cdot g_i = 0$, $i = 1,\ldots,r$.

We shall be particularly interested in the case $M = k[\Lambda]$. Since $k[Z^n] \simeq k[t_1,t_1^{-1},\ldots,t_n,t_n^{-1}]$ where $t_1,\ldots,t_n$ are the standard generators of $Z_+^n \subseteq Z^n$, we may identify $k[\Lambda]$ with a sub k-algebra of $k[t_1,t_1^{-1},\ldots,t_n,t_n^{-1}]$. If $\underline{f} \in F_o \simeq Z_+^r$, put $|\underline{f}|_j = \sum_{i=1}^{r} f_i a_{ij}$,

$j = 1,\dots,n$ where $\underline{a}_i = (a_{i1},\dots,a_{in}) \in Z^n$, and let $|\underline{f}| = (\underline{f}_1,\dots,\underline{f}_n) \in Z^n$. Since for $(\underline{f},\underline{g}) \in F_o \underset{\Lambda}{\times} F_o$, $|\underline{f}| = |\underline{g}|$, we may without confusion put $|(\underline{f},\underline{g})| = |\underline{f}| = |\underline{g}| \in Z^n$. Denote by $\underline{t}^{|(\underline{f},\underline{g})|}$ the product $\prod_{j=1}^{n} t_j^{|\underline{f}|_j} \in k[Z^n]$.

Given a derivation $D \in \ker\{Der(F_1,k[\Lambda]) \to Der(F_2 k[\Lambda])\}$ then the map $D_o: F_1 \to k[Z^n]$ defined by $D_o((\underline{f},\underline{g})) = \underline{t}^{-(\underline{f},\underline{g})} \cdot D((\underline{f},\underline{g}))$ factorizes through the surjection $F_1 \twoheadrightarrow J$, and we obtain a commutative diagram of maps

$$\begin{tikzcd} F_1 \arrow[rr, "D_o=t^{-\|\;\|}\cdot D"] \arrow[dr, "\eta"'] & & k[Z^n] \\ & J \arrow[ur, "D_1"'] & \end{tikzcd}$$

where D_1 is a homomorphism of the abelian group J into the abelian group underlying the k-module $k[Z^n]$.

Conversaly, if D_1 is a homomorphism, then $D = \underline{t}^{\|\;\|}\cdot D_o$ is a derivation contained in $\ker\{Der(F_1,k[\Lambda]) \to Der(F_2,k[\Lambda])\}$ if and only if $t^{\|\;\|}\cdot D_o$ maps $F_o \underset{\Lambda}{\times} F_o = F_1$ into $k[\Lambda] \subseteq k[Z^n]$.

Now, J is a free abelian group. Fix a basis $\{j_1,\dots,j_m\}$. Then D_1 is determined by a sequence of rank $J = m$ elements of $k[Z^n]$. Therefore we have an isomorphism

$$\begin{aligned} &\ker\{Der(F_1,k[\Lambda]) \to Der(F_2,k[\Lambda])\} \\ &\simeq \{(u_1,\dots,u_m) \in k[Z^n]^m \mid \underline{t}^{\|(\underline{f}_i,\underline{g}_i)\|}\cdot u_i \in k[\Lambda] \\ &\qquad \text{for all } i = 1,\dots,m, \text{ where } \eta(\underline{f}_i,\underline{g}_i) = j_i\} \end{aligned}$$

Moreover, $Der(F_o,k[\Lambda]) \simeq k(\Lambda)^r$, and any derivation $D' \in Der(F_o,k[\Lambda])$ is determined by the sequence $v_k = D'((\underbrace{0,\dots,1}_{k},\dots 0)) \in k[\Lambda]$, $k = 1, \dots, r$. The image D of D' in $Der(F_1,k[\Lambda])$ is defined by

$$D((\underline{f},\underline{g})) = D'(\underline{g})-D'(\underline{f}) = \sum_{\substack{k=1\\ g_k\neq 0}}^{r} (g_1,\dots,g_k-1,\dots,g_r)D'((0,\dots,\underset{k}{1},\dots,0))$$

$$-\sum_{\substack{k=1\\ f_k\neq 0}}^{r} (f_1,\dots,f_k-1,\dots,f_r)\cdot D'((0,\dots,\underset{k}{1},\dots,0)) = \sum_{k=1}^{r} \Big(\prod_{\ell=1}^{n} t_\ell^{\Sigma_{j=1}^{r} g_j a_{j\ell} - a_{k\ell}}\Big) v_k$$

The corresponding D_o is therefore given by

$$D_o((\underline{f}_i,\underline{g}_i)) = \sum_{\substack{k=1\\ g_k\neq 0}}^{r} (\prod_{\ell=1}^{n} t_\ell^{-a_{k\ell}})v_k - \sum_{\substack{k=1\\ f_k\neq 0}}^{r} (\prod_{\ell=1}^{r} t_\ell^{-a_{k\ell}})v_k .$$

Thus: $D_1(j_i) = \sum_{k=1}^{r} \delta_{ik}(\prod_{\ell=1}^{n} t_\ell^{-a_{k\ell}})\cdot v_k$ where $\delta_{ik} = 0$ if $j_{ik} = 0$, otherwise $\delta_{ik} = \epsilon\ \mathrm{sign}(j_{ik})$.

Summing up, we have got the following result

$$A^1(\Lambda;k[\Lambda]) \simeq \mathrm{coker}\{k[\Lambda]^r \overset{\Phi}{\to} \{(u_1,\dots,u_m) \in k[Z^n]^m | \underline{t}^{\|(f_i,g_i)\|}\cdot u_i \in k[\Lambda],\ i=1,\dots,m\}\}$$

where Φ is given by the matrix

$$\begin{bmatrix} \delta_{11}(\prod_{\ell=1}^{n} t_\ell^{-a_{1\ell}}),\dots & ,\delta_{1r}(\prod_{\ell=1}^{n} t_\ell^{-a_{r\ell}}) \\ \vdots & \\ \delta_{m1}(\prod_{\ell=1}^{n} t_\ell^{-a_{1\ell}}),\dots & ,\delta_{mr}(\prod_{\ell=1}^{n} t^{-a_{r\ell}}) \end{bmatrix}$$

Note that
$$\begin{aligned} \delta_{ik} &= 0 \quad \text{if} \quad f_{ik} = g_{ik} = 0 \\ \delta_{ik} &= -1 \quad \text{if} \quad f_{ik} \neq 0 \\ \delta_{ik} &= +1 \quad \text{if} \quad g_{ik} \neq 0 \end{aligned}$$

which is well defined since $f_{ik}\cdot g_{ik} = 0$ for all i,k.

As in [La 2,(5.1)] we find using the Leray spectral sequence:

$$A^2(\Lambda;k[\Lambda]) \simeq \ker\{A^1(F_1;k[\Lambda]) \overset{\to}{\to} A^1(F_2;k[\Lambda])\}.$$

Notice that if Λ is a free monoid, then $k[\Lambda]$ is a free k-algebra Moreover if M is any Λ-bi-module, then $\mathrm{Der}(\Lambda;M) \simeq \mathrm{Der}_k(k[\Lambda],M)$ for any monoid Λ. This suggests that there is a close correspondence between the cohomology of monoids and the cohomology of k-algebras.

This is the subject of the next two paragraphs.

<u>The case of profinite p-groups</u>. We shall need a slightly more refined theory to study profinite, or finite p-groups.

Let G be any group, p a prime and E any subset of G. Denote by $\Gamma_1(E) = \Gamma_1(E,G)$ the (normal) subgroup of G generated by the elements of the form $[g,e]$, $g\in G$, $e\in E$ and by the elements e^p, $e\in E$. Put for every $n \geq 2$, $\Gamma_n(E) = \Gamma_1(\Gamma_{n-1}(E))$, and put $\Gamma_n = \Gamma_n(G)$. Then $\subseteq \Gamma_{n+1} \subseteq \Gamma_n \subseteq \cdots \subseteq \Gamma_1 \subseteq G = \Gamma_0$ are all characteristic subgroups of G, and Γ_n/Γ_{n+1} are elementary abelian p-groups, i.e. F_p-vectorspaces.

Moreover, given any subcategory $\underline{c}$ of $\underline{gr}$, the correspondences $G \to \Gamma_n(G)$ and $G \to G/\Gamma_n(G)$, defines functors

$$\Gamma_n : \underline{c} \to \underline{gr}$$
$$id/\Gamma_n : \underline{c} \to \underline{gr}\ ,$$

respectively.

In particular if $\underline{c} = \underline{gr}$, the image of id/Γ_n is a subcategory $\underline{gr}_n$ of $\underline{gr}$ on which Γ_n is trivial, and the restriction of id/Γ_n to $\underline{gr}_{n+1}$ is a functor

$$id/\Gamma_n : \underline{gr}_{n+1} \to \underline{gr}_n\ , \qquad n \geq 1.$$

<u>Definition (1.1.8)</u>. Given a prime p, we shall let Pro-p-$\underline{gr}$ be the category for which the objects are sequences $\mathcal{G} = \{G_n\}_{n\geq 1}$ of finite p-groups $G_n \in \text{ob } gr_n$ together with homomorhisms $\eta_n^{n+1} : G_{n+1} \to G_n$ inducing isomorphisms $G_{n+1}/\Gamma_n(G_{n+1}) \simeq G_n$. If $\mathcal{G} = \{G_n\}$ and $\mathcal{G}' = \{G_n'\}$ are two objects of pro p-$\underline{gr}$, a morphism $\Psi: \mathcal{G} \to \mathcal{G}'$ is a sequence $\Psi = \{\psi_n\}_{n\geq 1}$ of morphisms $\psi_n : G_n \to G_n'$ making all relevant diagrams commute.

<u>Remark</u> (1.1.9). For the general notions of procategory or pro group, see [S 1].

Note that amalgamated sums and fibered products exist in pro p-$\underline{gr}$ and are defined at each level $\underline{gr}_n$, $n \geq 1$.

There is a canonical funcor:

$$\mathrm{id}/\Gamma\colon \underline{\mathrm{gr}} \to \mathrm{pro}\ \mathrm{p}\text{-}\underline{\mathrm{gr}}$$

defined by $(\mathrm{id}/\Gamma)(G) = \{G/\Gamma_n(G)\}_{n\geq 1}$. One checks easily that this functor maps free groups to free objects in pro p-gr. In particular $(\mathrm{id}/\Gamma)(F(S)) = F_p(S) = F(S)$ is projective as an object of pro p-gr. The full subcategory of pro p-gr generated by the objects of the form (S) should be called free pro p-gr, shortened to pro free.

If G and P are pro p-groups, an action of G on P is a sequence $\eta = \{\eta_n\}_{n\geq 1}$ of homomorphisms $\eta_n\colon G_n \to \mathrm{Aut}(P_n)$ making the diagrams

$$\begin{array}{ccc} G_{n+1} & \xrightarrow[\eta_{n+1}]{} & \mathrm{Aut}(P_{n+1}) \\ \downarrow & & \downarrow \\ G_n & \xrightarrow[\eta_n]{} & \mathrm{Aut}(P_n) \end{array}$$

commute.

If M is a G-bi-module, and if $E \subseteq M$ is a subset, put $\Gamma_1(E) = \Gamma_1(E,G)$ = submodule of M generated by the elements of the form gm-m and m-mg where $g \in G$, $m \in E$. By induction we define $\Gamma_n = \Gamma_1(\Gamma_{n-1}(E),G)+\Gamma_1(\Gamma_{n-2}(E),\Gamma_1(G))+\cdots+\Gamma_1(E,\Gamma_{n-1}(G))$.

Notice that $\subseteq \Gamma_{n+1}(E) \subseteq \Gamma_n(E) \subseteq\cdots\subseteq M$ is a filtration of M by G-bi-invariant k-submodules. We shall in particular be interested in the submodules $\Gamma_n = \Gamma_n(M)$. The motivation for this filtration is the following. If $D \in \mathrm{Der}(G,M)$ is a derivation, then D induces a derivation $G/\Gamma_n(G) \to M/\Gamma_n(M)$.

Given a pro-p-group G, a G-bi-module is then a sequence $M = \{M_n\}_{n\geq 1}$ of G_n-bi-modules M_n, together with homomorphisms $\rho_n^{n+1}\colon M_{n+1} \to M_n$ inducing isomorphisms of G_{n+1}-bi-modules $M_{n+1}/\Gamma_n(M_{n+1}) \approx M_n$. Let G-bi-mod be the category of -bi-modules.

Given a group G there is then a functor

$$\mathrm{id}/\Gamma\colon G\text{-bi-}\underline{\mathrm{mod}} \to (\mathrm{id}/\Gamma)(G)\text{-bi-}\underline{\mathrm{mod}}$$

defined by $(\mathrm{id}/\Gamma)(M) = \{M/\Gamma_n(M,G)\}_{n\geq 1}$, matching the functor $\mathrm{id}/\Gamma\colon \underline{\mathrm{gr}} \to \mathrm{pro}\ \mathrm{p}\text{-}\underline{\mathrm{gr}}$.

Now, as above we may define the notion of normal morphism of pro-p-groups. Moreover, given any morphism of pro-p-groups $\delta : P \to G$ we define the category P-pro-p-$\underline{gr}/G$ as usual. The objects of the form

$$(4) \qquad \begin{tikzcd} P \arrow[rr, "\delta'"] \arrow[dr, "\delta"'] & & Q \arrow[dl, "\phi"] \\ & G & \end{tikzcd}$$

where $Q = P * F(S)$ and δ' is the canonical morphism, will be called P-free. The full subcategory of P-pro-$\underline{gr}/G$ generated by the P-frees will be denoted by P-$\underline{free}/G$.

If δ is a normal morphism, we may, as above consider the category n-P-pro $\underline{gr}/G$ of diagrams where δ' is normal and ϕ is a morphism of normal morphisms. If the object (4) is P-free in the sense above, and if δ is normal, then the object

$$\begin{tikzcd} P \arrow[rr, "\bar{\delta}"] \arrow[dr] & & \bar{Q} \arrow[dl] \\ & G & \end{tikzcd}$$

defined by

$$\bar{Q} = \{P_n * F(S)_n \mid \{fpf^{-1}\phi(f)(p) \mid f \in F(S)_n, p \in P_n\} \cdot \Gamma_n(P_n * F(S)_n)\}_{n \geq 1}$$

is easily seen to be a free object of n-P-pro $\underline{gr}/G$. The full subcategory of n-P-pro $\underline{gr}/G$ generated by these objects will be denoted by n-P-$\underline{free}/G$.

Now, given a morphism (resp. normal morphism) $P \to G$ of pro p-groups and a G bi-module M, there is a natural functor

$$\mathrm{Der}_P(-,M) : P\text{-}\underline{\mathrm{free}}/G \to k\text{-}\underline{\mathrm{mod}}$$

$$(\text{resp. } \mathrm{Der}\ (-,M) : \text{n-}P\text{-}\underline{\mathrm{free}}/G \to k\text{-}\underline{\mathrm{mod}})$$

defined by:

$$\mathrm{Der}\left(\left(\begin{tikzcd} P \arrow[rr] \arrow[dr] & & Q \arrow[dl] \\ & G & \end{tikzcd}\right), M\right) = \mathrm{Der}_P(Q,M)$$

where $\mathrm{Der}_P(Q,M)$ is the k-module of sequences $D = \{D_n\}_{n \geq 1}$ of derivations $D_n \in \mathrm{Der}_{P_n}(Q_n, M_n)$ commuting with the morphisms $\eta_n^{n+1} : Q_{n+1} \to Q_n$ and $\rho_n^{n+1} : M_{n+1} \to M_n$.

Notice that when M is eventually constant, i.e. when $\rho_n^{n+1} : M_{n+1} \to$ $\to M_n = M$ are isomorphisms for all $n > N$, then $\mathrm{Der}_G(Q, \ ) =$ $= \varinjlim_n \mathrm{Der}(Q_n, M)$.

<u>Definition (1.1.10)</u>. The algebra cohomology of the morphism (resp. the normal morphism) $\rho : P \to G$ of pro p-groups with values in M, is the graded k-module

$$A^\bullet(P, G; M) = \varprojlim_{(P\text{-}\underline{\text{free}}/G)^o}{}^{(\bullet)} \mathrm{Der}_P(-. M)$$

$$(\text{resp: } A_n^\bullet(P, G; M) = \varprojlim_{(n\text{-}P\text{-}\underline{\text{free}}/G)^o}{}^{(\bullet)} \mathrm{Der}(-, M))$$

As in the case of groups we shall use the shorthand $A^\bullet(G, M)$ for $A^\bullet(\{1\}, G; M)$.

<u>Remark (1.1.11)</u>. With this definition there are propositions analogous to (1.1.3) and (1.1.4). The changes we will have to make in the statments are pretty obvious. Observe that (1.1.3) is a categorical statement depending only upon the existence of fibered products and free objects in the relevant categories. (1.1.4) follows as before from (1.1.3) by streightforward computation.

<u>Corollary (1.1.12)</u> (Tate). If G is a pro-p-grous then

(i) $d = \dim_{F_p} A^0(G, F_p)$ is the minimum number of generators of G as a pro-p-group.

(ii) $r = \dim_{F_p} A^1(G, F_p)$ is the minimum number of relations in a presentation of G.

<u>Proof</u>. This follows from (1.1.4) and (1.1.11). In fact $A^0(G,F_p) \simeq \mathrm{Der}(G,F_p) \simeq \mathrm{Hom}_{F_p}(G/\Gamma_1(G),F_p)$, and $A^1(G,F_p) \simeq \mathrm{Hom}_F(J,F_p)/\mathrm{Der} = \varinjlim_k \mathrm{Hom}(J_k/\Gamma_1(J_k,F_k),F_p)$ where $F = F(S)$ is a free profinite p-group, $\pi\colon F \to G$ is a surjection and $J = \ker \pi$. Remember that $\Gamma_1(J_k,F)$ = group generated by the elements $[f,j]$ and j^p where $f \in F_k$ and $j \in J_k$. Q.E.D.

(1.2) <u>Cohomology of algebras</u>

In this paragraph we shall sketch how to generalize the ordinary cohomology of commutative algebras, see [An], [Q] and [La], to the case of noncommutative algebras. There are no surprises, but since we shall need the formalism later on we shall never the less give all nessecary definitions and state the theorems we need in the sequel.

Consider the category of k-algebras for which k is central, k-<u>alg</u>. A free k-algebra is simply a tensor algebra on a free k-module. If T is any set $F(T)$ denote the tensor algebra on the k-module $k^{(T)} = \coprod_T k$. $F(T)$ is obviously a free k-algebra, and a free object of k-<u>alg</u>.

Given any k-algebra S we shall consider the k-algebras of the form $S\coprod_k F(T)$ i.e. the categorical direct sum of S and $F(T)$ in k-<u>alg</u>. (This is not the tensor S-algebra of $S^{(T)} = \coprod_T S$, as one might hope.)

Given a morphism $\delta\colon S \to A$ of k-algebras, we may consider the category S-<u>alg</u>/A of commutative diagrams of k-algebras

$$\begin{tikzcd} S \arrow[rr, "\delta'"] \arrow[dr, "\delta"'] & & B \arrow[dl, "\phi"] \\ & A & \end{tikzcd}$$

The full subcategory of S-<u>alg</u>/A generated by those objects for which $B = S\coprod_k F(T)$ and δ' is the canonical morphism, is denoted by S-<u>free</u>/A.

For any A-bi-module M, define

$$\mathrm{Der}_S(A,M) = \{D \in \mathrm{Hom}_k(A,M)/\forall a_1,a_2 \in A, D(a_1 \cdot a_2) = a_1 D(a_2)+D(a_1)a_2, \rho \circ D=0\}.$$

With this done, we copy the construction of the cohomology of (1.1),

<u>Definition (1.2.1)</u>. The algebra cohomology of the morphism $S \to A$ with values in M is the graded k-module

$$A^{\bullet}(A,A;M) = \varprojlim_{S\text{-}\underline{\mathrm{free}}/A}{}^{(\cdot)} \mathrm{Der}_S(-.M).$$

<u>Proposition (1.2.2)</u>. Let $S \amalg_k F(T) \to A$ be a surjective morphism of S-algebras. Put $F_o = S \amalg_k F(T)$ and consider the simplicial S-algebra

$$F_{\bullet}: A \leftarrow\!-\!- F_o \leftleftarrows F_o \times_A F_o \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, F_o \times_A F_o \times_A F_o \,\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}\, \cdots$$

then there is a spectral sequence with

$$E_2^{pq} = H^p(\overbrace{A^q(S,F_{\bullet};M)})$$

converging to $A^{\bullet}(S,A;M)$.

<u>Proposition (1.2.3)</u>. (i) Any short exact sequence of A-bi-modules induces a long exact sequence in cohomology.

(ii) $A^o(S,A;M) = \mathrm{Der}_S(A,M)$

$A^1(S,A;M) = \mathrm{Hom}_{F_o}(J,M)/\mathrm{Der}$

where J is the kernel of any surjective S-morphism on $\delta: S_k F(T) \to A$.

We also copy the definitions of pro-category and the corresponding cohomology, for the case of k-algebrs.

The category of augmented k-algebras is the category $k\text{-}\underline{\mathrm{alg}}/k$. Given an object $k \overset{\delta}{\to} A \overset{\rho}{\to} k$, the ideal $\underline{m} = \ker \rho$ is called the augmentation ideal. The powers $\underline{m}^{n+1}$ of the augumentation ideal is going to play the role of Γ_n, see (1.1). In particular we define the functor

$$id/\underline{m}^{n+1}: k\text{-}\underline{alg}/k \to k\text{-}\underline{alg}/k$$

by:

$$id/\underline{m}^{n+1}(k \to A \to k) = k \to A/\underline{m}^{n+1} \to k$$

The image of this functor is denoted by $(k\text{-}\underline{alg}/k)_n$, $n \geq 0$. There is an obvious restriction functor

$$id/\underline{m}^{n+1}: (k\text{-}\underline{alg}/k)_{n+1} \to (k\text{-}\underline{alg}/k)_n$$

<u>Definition (1.2.4)</u>. We shall denote by pro-k-<u>alg</u> the category for which the objects are sequences $\mathcal{A} = \{A_n | n \geq 0\}$ of augemented k-algebras A_n, together with morphisms $\eta_n^{n+1}: A_{n+1} \to A_n$ inducing isomorhisms

$$A_{n+1}/\underline{m}^{n+1}(A_{n+1}) \simeq A_n.$$

A morphism $\Phi: \mathcal{A} \to \mathcal{B}$, where $\mathcal{A} = \{A_n\}_{n\geq 0}$ and $\mathcal{B} = \{B_n\}_{n\geq 0}$ are pro k-algebras is a sequence $\Phi = \{\phi_n\}_{n\geq 0}$ of morphisms of augmented k-algebras $\phi_n: A_n \to B_n$, making all relevant diagrams commute.

There is an obvious functor

$$\Gamma: k\text{-}\underline{alg}/k \to \text{pro } k\text{-}\underline{alg}$$

defined by $\Gamma(A) = \{A/\underline{m}^{n+1}(A)\}_{n\geq 0}$. Since the free k-algebra $F(T)$ has a natural augmentation, mapping T to 0, $\mathcal{F}(T) = \Gamma(F(T))$ is defined. It is easy to see that $\mathcal{F}(T)$ is a projective object of pro k-<u>alg</u>. The full subcategory of pro k-<u>alg</u> generated by these objects is called free pro k-<u>alg</u>.

If $k \to A \to k$ is an augemented k-algebra, and M is an A bi--module, put for any subset E of A, $\Gamma_1(E)$ = ideal generated by $\underline{m}\cdot E + E\cdot\underline{m}$ in A. By induction $\Gamma_n(E) = \Gamma_1(\Gamma_{n-1}(E))$.

Now, given a pro k-algebra $\mathcal{A} = \{A_n\}_{n\geq 1}$ an $\mathcal{A}$-bi-module is a sequence $\mathcal{M} = \{M_n\}_{n\geq 1}$ of A_n-bi-modules M_n, together with morphisms of A_{n+1}-bi-modules $M_{n+1}/\Gamma_n(M_{n+1}) \simeq M_n$.

Put:

$$\mathrm{Der}_k(A,M) = \{D=\{D_n\}_{n\geq 1} \mid D_n \in \mathrm{Der}(A_n,M_n) \text{ s.t. } D_{n+1} \circ \rho_n^{n+1} = \eta_n^{n+1} \circ D_n,\ \forall\, n\geq 1\},$$

and consider the functor:

$$\mathrm{Der}_k(-,M): (\text{free pro } k\text{-}\underline{\mathrm{alg}}/A)^{\circ} \to k\text{-}\underline{\mathrm{mod}}$$

defined as in (1.1).

<u>Definisjon (1.2.5)</u>. The algebra cohomology of the pro k-algebra A with values in M is the graded k-module

$$A^{\bullet}(k,A;M) = \varprojlim_{(\text{free pro } k\text{-}\underline{\mathrm{alg}}/A)^{\circ}}{}^{(\bullet)} \mathrm{Der}(-,M)$$

<u>Remark (1.2.6)</u>

It is pretty safe to leave to the reader to state and prove the analogies of (1.2.2) and (1.2.3) in the pro-situation. Notice, in particular, that the Corollary (1.1.12) properly modifed holds in the pro k-algebra case.

(1.3) <u>Relations between the cohomology of groups and the cohomology of the corresponding group-algebras</u>

Let G be a group and let M be a G-bi-module. M is by definition a k(G)-bi-module, and we may consider the cohomology of the groups $A^{\bullet}(G,M)$ and the cohomology of the corresponding group-algebra $A^{\bullet}(k,k(G);M)$. Since $A^{0}(G,M) = \mathrm{Der}(G,M) = \mathrm{Der}_k(k(G),M) = A^{0}(k,k(G);M)$ it is reasonable to believe that there exist some kind of relationship between the two types of cohomology. In fact, we shall show that there is a natural spectral sequence relating them.

<u>Remark (1.3.1)</u>. Observe that the construction part of this paragraph works equally well for pro groups and monoids as for groups. To avoid boring repetitions, we shall not insist upon the

obvious changes in notations etc., assuming that the reader will see what should be done.

Observe also that there is a commutative theory, as well as a noncommutative theory, see (1.1).

Consider the functor

$$i:\ \underline{\mathrm{free}}/G \to k\text{-}\underline{\mathrm{alg}}/k(G)$$

defined by $i(F \to G) = k(F) \to k(G)$. Given an object $\delta: F \to G$ of $\underline{\mathrm{free}}/G$ we may also consider the functor

$$j:\ k\text{-}\underline{\mathrm{free}}/k(F) \to k\text{-}\underline{\mathrm{free}}/k(G)$$

defined by $j(\bar{F} \xrightarrow{\delta} k(F)) = \bar{F} \xrightarrow{\rho\circ\delta} k(G)$.

j: induces a morphism of complexes

$$\begin{array}{l} C^{\bullet}(k\text{-}\underline{\mathrm{free}}/k(G),\ \mathrm{Der}_k(-,M)) \\ \downarrow \\ C^{\bullet}(k\text{-}\underline{\mathrm{free}}/k(F),\ \mathrm{Der}_k(-,M)) = C^{\bullet}(\delta) \end{array}$$

The last complex is a contravariant functor defined on $\underline{\mathrm{free}}/G$, and we may consider the double complex $C^{\bullet}((\underline{\mathrm{free}}/G)^{o}, C^{\bullet}(k\text{-}\underline{\mathrm{free}}/k(-), \mathrm{Der}_k(-,M)))$. Since $H^{0}(C^{\bullet}(k\text{-}\underline{\mathrm{free}}/k(F), \mathrm{Der}_k(-,M)) = \mathrm{Der}_k(k(F),M) = \mathrm{Der}(F,M)$ there are canonical morphisms of complexes:

$$\begin{array}{l} C^{\bullet}(\underline{\mathrm{free}}/G, \mathrm{Der}(-,M)) \\ \downarrow \\ C^{\bullet}(\underline{\mathrm{free}}/G, C^{\bullet}(k\text{-}\underline{\mathrm{free}}/k(-), \mathrm{Der}_k(-,M))) \\ \uparrow \\ C^{\bullet}(k\text{-}\underline{\mathrm{free}}/k(G), \mathrm{Der}_k(-,M)). \end{array}$$

Proposition (1.3.2). For all groups G and any G-bi-module M, we have

$$\varprojlim_{(\underline{\mathrm{free}}/G)^{o}}{}^{(p)}\ C^{\bullet}(k\text{-}\underline{\mathrm{free}}/k(-), \mathrm{Der}_k(-,M))$$

$$= \begin{cases} 0 & \text{for } p > 1 \\ C^{\bullet}(k\text{-}\underline{\mathrm{free}}/k(G), \mathrm{Der}_k(-,M)) & \text{for } p = 0. \end{cases}$$

Assume the proposition proved, then the first spectral sequence of the double complex above converges to the cohomology of the double complex, which by the second spectral sequence is seen to be isomorphic to $A^{\bullet}(k,k(G);M)$. This implies,

<u>Corollary (1.3.3)</u>. For any group G, and any G-bi-module M, there is a spectral sequence with

$$E_2^{p,q} = \varprojlim_{(\underline{\text{free}}/G)^{o}}{}^{(p)} A^q(k,k(-);M)$$

converging to $A^{\bullet}(k,k(G);M)$.

In particualar there is an edge homomorphism

$$A^{\bullet}(G,M) \to A^{\bullet}(k,k(G);M).$$

<u>Corollary (1.3.4)</u>. If G is a pro-p-group, and M is a G-bi-module, there are natural isomorphisms

$$A^{\bullet}(G,M) \simeq A^{\bullet}(F_pF_p(G);M).$$

<u>Proof</u>. This follows from the fact that if F is a free pro-p-group, then $F_p(F)$ is a free-pro-F_p-algebra. Therefore the spectral sequence of (1.3.3) degenerates, and the result follows from $A^o(k,k(-);M) = Der_k(-,M)$.

Q.E.D.

There is an interesting consequence in the commutative case, namely:

<u>Corollary (1.3.5)</u>. For any abelian group G and any G-bi-module M there is a canonical isomorphism

$$A^{\bullet}_{com}(G,M) \simeq A^{\bullet}_{com}(k,k(G);M)$$

<u>Proof</u>. Due to (1.3.3) we just have to prove that for any $F \to G$, $F \simeq Z^n$ being a free abelian group $A^q(k,k(F);M) = 0$ for $q \geq 1$. But this follows from the fact that $k(F) \simeq k[t_1,t_1^{-1},\ldots,t_n,t_n^{-1}] \simeq k[t_1,\ldots,t_n]_{\{t_1\ldots\ldots t_n\}}$ is a localization of a free k-algebra.

Therefore $A^q(k,k(F);M) = A^q(k,k[t_1,\dots,t_n];M)_{\{t_1,\dots,t_n\}} = 0$ for $q > 1$, see [An].

Q.E.D.

Proof of the proposition (1.3.2). Notice that the functor P: free/G $\to$ k-mod defined by

$$C^p(F\to G) = C^p(k\text{-}\underline{\text{free}}/k(F), \mathrm{Der}_k(-,M))$$

is a product of the functors

$$E = E\left(\begin{tikzcd} \bar F_o \arrow[r, "\psi_1"] \arrow[dr, "\delta_o"'] & \bar F_1 \arrow[r] \arrow[d] & \cdots \arrow[r, "\psi_p"] \arrow[dl] & \bar F_p \arrow[dll, phantom, "\delta_p"] \\ & k(G) & & \end{tikzcd}\right): \underline{\text{free}}/G \to k\text{-}\underline{\text{mod}}$$

defined for every string of p composable morphisms $(\psi_1,\dots,\psi_p)$ in k-free/k(G), by

$$E(F\to G) = \prod_{\begin{tikzcd}[column sep=small] \bar F_p \arrow[rr, "\phi"] \arrow[dr, "\delta_p"'] & & k(F) \arrow[dl] \\ & k(G) & \end{tikzcd}} \mathrm{Der}_k(\bar F_o, M).$$

It suffices therefore to prove that

$$\varprojlim_{(\underline{\text{free}}/G)^o}{}^{(p)} E = 0 \qquad \text{for } p \geq 1$$

$$\varprojlim_{(\underline{\text{free}}/G)^o} E = \mathrm{Der}_k(\bar F_o, M).$$

Put $\mathrm{Der}_k(\bar F_o, M) = N$, and observe that E is the restriction to free/G of a functor defined on gr/G. Let $\delta: F \to G$ be an object of free/G, such that δ is surjective, and consider the semi-simplicial group object

$$F_\bullet: G \longleftarrow F \leftleftarrows F \underset{G}{\times} F \;\overset{\leftarrow}{\leftleftarrows}\; F \underset{G}{\times} F \underset{G}{\times} F \;\overset{\leftarrow}{\underset{\leftarrow}{\leftleftarrows}}\; \cdots$$

Apply the extended functor E to this semi-simplicial object, and obtain the complex

$$E(F_\bullet):\ \left(\bar F_p \xrightarrow{\Pi N} k(F)\right)_{/k(G)} \rightrightarrows \left(\bar F_p \xrightarrow{\Pi N} k(F\times_G F)\right)_{/k(G)} \overrightarrow{\underrightarrow{\to}} \cdots$$

We want to prove that this complex is ascyclic. This amounts to proving that the semi-simplicial set

$$\bar K = \operatorname{Mor}_{k\text{-}\underline{alg}/k(G)}(\bar F_p, k(F_\bullet))$$

is ascyclic.

Lemma (1.3.6). Consider the simplicial abelian groups

$$K : k(F) \leftleftarrows k(F\times_G F) \overleftarrow{\underleftarrow{\leftarrow}} \cdots\cdot$$

and

$$L : k(F) \leftleftarrows k(F) \underset{k(G)}{\times} k(F) \overleftarrow{\underleftarrow{\leftarrow}} \cdots$$

(i) The canonical morphism of simplicial abelian groups $\rho\colon K \to L$, is surjective.

(ii) The kernel of ρ, which is a simplicial abelian group, is homotopically trivial.

(iii) Let $\pi\colon K \to k(G)$, $\pi_o\colon L \to k(G)$ be the canonical morphisms of simplicial groups, and pick an element $\bar x \in k(G)$. Then $K(\bar x) = \pi^{-1}(\bar x)$ and $L(\bar x) = \pi_o^{-1}(\bar x)$ are Kan complexex and the restriction of ρ,

$$\rho_{\bar x}\colon K(\bar x) \to L(\bar x)$$

is Kan-fibration with fiber $\ker \rho$.

Proof of lemma. Simplicial groups are Kan complexes and surjective morphisms of simplicial groups are Kan-fibrations.
Therefore K, L and the constant simplicial group $k(G) \leftleftarrows k(G) \underset{k(G)}{\times} k(G) \overleftarrow{\underleftarrow{\leftarrow}} \cdots$ denoted by $k(G)$ are Kan complexes, as are the fibers $K(\bar x)$ and $L(\bar x)$. This proves (iii).

To prove (i), pick any element $\omega = (\Sigma_{i=1}^{n_o}\alpha_i f_i, \Sigma_{j=1}^{n_1}\beta_j g_j, \cdots, \Sigma_{k=1}^{n_p}\gamma_k h_k)$

of $k(F)\times k(F)\times\cdots\times k(F) = L_p$, where we assume all coefficients $\alpha_i, \beta_j, \cdots, \gamma_k$ occuring are $\neq 0$. If $\omega = 0$ then obviously $\omega \in \operatorname{im}\rho$. This is the case iff $\|\omega\| = \{n_0, \cdots, n_p\} = (0,0,\cdots,0)$. Now we shall show that if $\|\omega\| > 0$ in the obvious ordered set Z_+^{p+1}, then we may find ω_1, and ω_2 such that $\omega = \omega_1 + \omega_2$, where $\omega_1 \in \ker\rho$ and $\|\omega_2\| < \|\omega\|$. The assertion follows then by induction on $\|\omega\|$.
So assume $\|\omega\| > 0$. We may then assume that $n_0 \geq 1$, therefore that $\alpha_1 \neq 0$. Let $\{f_1, f_{i_2}, \cdots, f_{i_s}\}$, $\{g_{j_1}, g_{j_2}, \cdots, g_{j_t}\}, \cdots \cdots$, $\{h_{k_1}, h_{k_2}, \cdots, h_{k_u}\}$ be the subsets formed by the elements of $\{f_i\}, \{g_j\}, \cdots, \{h_k\}$, respectively, mapped to $\delta(f_1)$ by δ (i.e. $\delta(f_1) = \delta(f_{i_2}) = \cdots = \delta(f_{i_s}) = \delta(g_{j_1}) = \delta(g_{j_2}) = \cdots = \delta(g_{j_t}) = \cdots = \delta(h_{k_1}) = \delta(h_{k_2}) = \cdots = \delta(h_{k_u})$). Suppose one of these subsets is empty, say $\{g_{j_1}, \cdots, g_{j_2}\} = \emptyset$. Then since $\delta'(\Sigma\alpha_i f_i) = \delta'(\Sigma\beta_j g_j)$, where $\delta'\colon k(F) \to k(G)$ is induced by δ, we must have $\Sigma_{\ell=1}^{s}\alpha_{i_\ell} = \Sigma_{\emptyset}\beta_j = 0$. This is a concequence of $k(G)$ being a free k-module.

In this case put $\bar{g}_\ell = f_1$ for $\ell = 1, \cdots, s$. If a subset other than $\{f_1, f_{i2}, \ldots, f_{is}\}$, say $\{h_{k_1}, \cdots, h_{k_u}\}$, is not empty, put $\bar{h}_\ell = h_{k_1}$, $\ell = 1, \cdots, s$. But let $\bar{f}_\ell = f_{i\ell}$ for $\ell = 1, \cdots, s$. Consider now the element $\omega_1 = (\Sigma_{\ell=1}^{s}\alpha_{i_\ell}\bar{f}_\ell, \Sigma_{\ell=1}^{s}\alpha_{i_\ell}\bar{g}_\ell, \cdots, \Sigma_{\ell=1}^{s}\alpha_{i_\ell}\bar{h}_\ell) \in L_p$. Obviously $\omega_1 = \rho(\Sigma_{\ell=1}^{s}\alpha_{i_\ell}(\bar{f}_\ell, \bar{g}_\ell, \cdots, \bar{h}_\ell)) \in \operatorname{im}\rho$ and $\|\omega - \omega_1\| < \|\omega\|$.
This proves (i), and we are left with (ii).
Consider

$$U_p = (\ker\rho)_p = \{\Sigma_{i=1}^{r}\alpha_i(f_{0i}, f_{1i}, \cdots, f_{pi}) \mid (f_{0i}, f_{1i}, \cdots, f_{pi}) \in F \underset{G}{\times} F \underset{G}{\times}\cdots\underset{G}{\times} F,\ \Sigma_{i=1}^{r}\alpha_i f_{ki} = 0 \text{ for } k = 0, \cdots, p\}.$$

Any element of U_p may be written in the following way

$$\sum_{g\in G}\ \sum_{\{i \mid g=\delta(f_{0i})\}} \alpha_i(f_{0i}, f_{1i}, \cdots, f_{pi})$$

Therefore we must have

$$\sum_{\{i|\delta(f_{ki})=g\}} \alpha_i f_{ki} = 0 \quad \text{for} \quad k = 0,\cdots,p \quad \text{and} \quad g \in G.$$

In particular $\sum_{\{i|g=\delta(f_{ki})\}} \alpha_i = 0$ for $k = 0,\cdots,p$ and $g \in G$.

Let $\sigma\colon G \to F$ be a set-theoretical section of $\delta\colon F \to G$. Consider the map

$$H_p\colon U_p \to U_{p+1}$$

defined by $H_p(\Sigma_{i=1}^r \alpha_i(f_{oi},\cdots,f_{pi})) = \Sigma_{i=1}^r \alpha_i(f_{oi},\cdots,f_{pi},\sigma(\delta(f_{oi})))$ which is weell defined since

$$\Sigma_{i=1}^r \alpha_i(\sigma(\delta(f_{oi}))) = \sum_{g\in G} \sum_{\{i|g=\delta(\sigma(\delta(f_{oi})))\}} \alpha_i(\sigma(\delta(f_{oi})) = \sum_{g\in\Gamma} \sum_{\{i|g=\delta(f_{oi})\}} \alpha_i \sigma_{(g)} = 0$$

H_p is additiv, and we shall see that $\{H_p\}_{p\geq 0}$ is a contracting homotopy of U. In fact $H_{p-1}(d(\Sigma_{i=1}^r \alpha_i(f_{oi},\cdots,f_{pi}))) =$

$$H_{p-1}(\Sigma_{k=0}^p(-1)^k d_k(\Sigma_{i=1}^r \alpha_i(f_{oi},\cdots,f_{pi}))) = H_{p-1}(\Sigma_{k=0}^r(-1)^k (\Sigma_{i=1}^r \alpha_i(f_{oi},\cdots,\hat{f}_{ki},\cdots,f_{pi}))$$

$$= \Sigma_{k=0}^p(-1)^k\ \Sigma_{i=1}^r \alpha_i(f_{oi},\cdots,\hat{f}_{ki},\cdots,f_{pi},\sigma(\delta(f_{oi}))) =$$

$$= \Sigma_{k=0}^p(-1)^k d_k(H_p((\Sigma_{i=1}^r \alpha_i(f_{oi},\cdots,f_{pi})))$$

$$= d\ H_p(\Sigma_{i=1}^r \alpha_i(f_{oi},\cdots,f_{pi})) - (-1)^{p+1} \Sigma_{i=1}^r \alpha_i(f_{oi},\cdots,f_{pi})$$

i.e. $H_{p-1}d - d\ H_p = (-1)^p\ \mathrm{id}$.

Q.E.D.

Proof of (1.3.2) continued. Let $\bar{F}_p$ be the free k-algebra on the set $\{x_1,\cdots,x_q\}$. Put $\bar{x}_i=\delta_p(x_i)$, $i = 1,\cdots,q$ where we recall that δ_p is the morphism $\bar{F}_p \to k(G)$, part of the definition of E above. Observe that the simplicial set $\bar{K}=\mathrm{Mor}(\bar{F}_p,K)$ is the carthesian product of $K(\bar{x}_i)$, $i = 1,\cdots,q$, and that $\bar{L} = \mathrm{Mor}(\bar{F}_p,L)$ is the carthesian product of $L(\bar{x}_i)$, $i = 1,\cdots,q$. Now, for each $i = 1,\cdots,q$, $L(\bar{x}_i)$ is the trivial simplicial set, i.e. if $L_o = L(\bar{x}_i)_o$ then $L(\bar{x}_i)_p = \underbrace{L_o\times\cdots\times L_o}_{p+1}$.

Therefore $L(\bar{x}_i)$ is homotopically trivial, which implies that $K(\bar{x}_i)$ is homotopically trivial, since U is. In particular $K(\bar{x}_i)$ is acyclic. By Eilenberg-Zilber the carthesian product of $K(\bar{x}_i)$ for $i = 1,\cdots,q$ is therefore acyclic, thus $\bar{K}$ is acyclic.

Using the Leray spectral sequence [La 2] we have proved that

$$\varprojlim_{(\underline{\text{free}}/G)^{\circ}} E = N$$

$$\varprojlim_{(\underline{\text{free}}/G)^{\circ}}{}^{(1)} E = 0$$

Replace G by the group $F \underset{G}{\times} \cdots \underset{G}{\times} F = F_p$, and obtain

$$\varprojlim_{(\underline{\text{free}}/F_p)^{\circ}}{}^{(1)} E = 0 \qquad \text{for } p \geq 0$$

therefore by the same spectral sequence,

$$\varprojlim_{(\underline{\text{free}}/G)^{\circ}}{}^{(2)} E = 0$$

etc.

This proves the proposition. Q.E.D.

<u>Remark (1.3.7)</u>. In the above proof we never used the group properties of the objects G, F etc. It is easy to see that the same results hold for progroups and monoids.

<u>Corollary (1.3.8)</u>. Let Λ be a monoid and let M be any Λ-bimodule. Then we have natural isomorphisms (in the commutative as well as in the non-commutative case)

$$A^i(\Lambda,M) \simeq A^i(k,k(\Lambda);M) \qquad i \geq 0.$$

<u>Proof</u>. If Γ is a free monoid, then $k(\Gamma)$ is a free k-algebra. Then use the monoid variant of (1.3.3) Q.E.D.

<u>Remark (1.3.9)</u>. If $\Lambda \subseteq Z^n$ is a commutative monoid, then (1.1.7) furnishes a calculation of $A^i(\Lambda,k(\Lambda))$ for $i = 1,2$. By

(1.3.8) this gives us $A^i(k,k(\Lambda);k(\Lambda))$ for $i = 1,2$. In particualar $A^1(\Lambda,k(\Lambda))$ is the tangent space of the formal moduli of the singularity $k(\Lambda)$.

(1.3.10) Appendix on the Betti-numbers of monoid algebras

Let Λ be a commutative monoid with cancellation law, i.e. such that $\lambda\cdot\mu = \lambda\cdot\mu'$ implies $\mu = \mu'$.
Let $A = k[\Lambda]$ and put $\underline{m} = \Lambda_+\cdot A$ where $\Lambda_+ = \Lambda\setminus\{1\}$. Assume $A/\underline{m} = k$, i.e. assume Λ has no non-trivial subgroups.
Put $\beta_i = \dim_k \operatorname{Tor}_i^A(k,k)$, the i-th Betti-number of $k[\Lambda]$.
In this appendix I shall show how to compute the β_i's using only the combinatorial properties of Λ_+,
Let Λ_+ be ordered as follows; $\lambda_1 \leq \lambda_2$ if and only if there exists a $\mu \in \Lambda$ such that $\mu\cdot\lambda_1 = \lambda_2$. There is a natural presheaf (projective system) defined on Λ_+,

$$F\colon \Lambda_+ \to \underline{Ab}$$

with

$$F(\lambda) = A$$

$$F(\lambda_1\leq\lambda_2)\colon F(\lambda_2) \to F(\lambda_1),\ \text{multiplication by}$$

$$\mu = \frac{\lambda_2}{\lambda_1}\ .$$

<u>Lemma 1</u>. $\varinjlim_{\Lambda_+} F = (\Lambda_+)\cdot A = \underline{m}$.

<u>Proof</u>. For every $\lambda \in \Lambda_+$, consider the morphism $\eta_\lambda\colon F(\lambda) \to A$, multiplication by λ. This defines a morphism $\eta\colon \varinjlim_{\Lambda_+} F \to \underline{m}$.
Given an element $\alpha \in \underline{m}$, then there is a unique representation $\alpha = \sum_{i=1}^N \alpha_i\cdot\lambda_i$, $\alpha_i \in k$, $\lambda_i \in \Lambda_+$. Consider α_i as an element of $F(\lambda_i)$ and let $\bar{\alpha}_i$ be the image of α_i in $\varinjlim_{\Lambda_+} F$. Define $\mu\colon \underline{m} \to \varinjlim_{\Lambda_+} F$ by $\mu(\alpha) = \sum_{i=1}^N \bar{\alpha}_i$. Then μ is an inverse of η.

Q.E.D.

<u>Lemma 2</u>. $\varprojlim_{\Lambda_+}^{(n)} F = 0$ for $n \geq 1$.

<u>Proof</u>. By [La 1,(1.1.4)] it is enough to show that F is coflabby (coflasque). Thus, let $\lambda \in \Lambda_+$ and suppose $\Lambda_1 \subseteq \{\lambda' \in \Lambda_+ | \lambda \leq \lambda'\}$ is such that if $\lambda' \in \Lambda_1$ and $\lambda' \leq \lambda''$ then $\lambda'' \in \Lambda_1$. F is coflabby if in this situation

$\varprojlim_{\Lambda_1} F \to \varprojlim_{\{\lambda' \in \Lambda_+ | \lambda \leq \lambda'\}} F = F(\lambda) = A$ is an injection.

However, the proof of Lemma 1 applies to show that $\varprojlim_{\Lambda_1} F =$

$= \{\frac{\lambda'}{\lambda} | \lambda' \in \Lambda_1\} \cdot A$ and that the morphism $\varprojlim_{\Lambda_1} F \to \varprojlim_{\{\lambda' \in \Lambda_+ | \lambda \leq \lambda'\}} F = A$

is the obvious inclusion. Therefore we are done. Q.E.D.

Consider the resolving complex $C_\bullet(\Lambda_+;-)$ for $\varprojlim_{\Lambda_+}$, see [R] or [La 1,(1.2)]. By Lemma 2, $C_\bullet(\Lambda_+,F)$ is an A-free resolution of the maximal ideal $\underline{m}$ of A. Therefore

$$\mathrm{Tor}_i^A(k,k) \simeq \begin{cases} k & i = 0 \\ H_{i-1}(C_\bullet(\Lambda_+;F) \otimes_A k) & i \geq 1 \end{cases}$$

Now $C_\bullet(\Lambda_+;F) \otimes_A k = C_\bullet(\Lambda_+;F \otimes_A k)$, therefore $H_{i-1}(C_\bullet(\Lambda_\bullet;F) \otimes_A k) =$
$= \varprojlim_{\Lambda_+}^{(i-1)} (F \otimes_A k)$.

Observe that the projective system $F \otimes_A k$ is isomorphic to $\prod_{\lambda \in \Lambda_+} k(\lambda)$, where $k(\lambda)$ is the projective system defined by:

$$k(\lambda)(\lambda') = \begin{cases} 0 & \text{if } \lambda' \neq \lambda \\ k & \text{if } \lambda' = \lambda \end{cases}$$

$k(\lambda)(\lambda' \leq \lambda'')$: $k(\lambda)(\lambda'') \to k(\lambda)(\lambda')$ is zero if $\lambda' \neq \lambda''$.

Put for any $\lambda \in \Lambda_+$,

$$\hat{\lambda} = \{\lambda' \in \Lambda_+ | \lambda' \leq \lambda\}$$

$$L(\lambda) = \{\lambda' \in \Lambda_+ | \lambda' \leq \lambda, \lambda' \neq \lambda\}.$$

It is easy to see that there are isomorphisms:

$$\varinjlim_{\Lambda_+}{}_{(n)}\, k(\lambda) \simeq \varinjlim_{\hat{\lambda}}{}_{(n)}\, k(\lambda) \qquad \text{for } n > 0.$$

In fact this follows from the existence of a $\amalg$-projective resolution of $k(\lambda)$ trivial outside of $\hat{\lambda}$, see [La 1,(1.2)]. Let $\underline{k}_\lambda$ be the constant projective system on λ defined by $\underline{k}_\lambda(\lambda') = k$, and let $\underline{k}'_\lambda$ be the sub projective system of $\underline{k}_\lambda$ defined by $\underline{k}'_\lambda(\lambda')=0$ if $\lambda' = \lambda$ and $\underline{k}'_\lambda(\lambda') = k$ if $\lambda' \neq \lambda$. Then there is an exact sequence of projective systems on λ

$$0 \to \underline{k}'_\lambda \to \underline{k}_\lambda \to k(\lambda) \to 0$$

As

$$\varinjlim_{\hat{\lambda}}{}_{(n)}\, \underline{k}_\lambda = \begin{cases} k & \text{for } n = 0 \\ 0 & \text{for } n \geq 1 \end{cases}$$

and since

$$\varinjlim_{\hat{\lambda}}{}_{(n)}\, \underline{k}'_\lambda \simeq \varinjlim_{L(\lambda)}{}_{(n)}\, \underline{k} \simeq H_n(E(\lambda);k) \qquad n > 0$$

where $\underline{k}$ is the constant projective system k on $L(\lambda)$, and where we denote by $E(\lambda)$ the simplicial set defined by the ordered set $L(\lambda)$, see [La 1,(1.1)], we obtain an exact sequence

$$0 \to \varinjlim_{\hat{\lambda}}{}_{(1)} k(\lambda) \to \varinjlim_{\hat{\lambda}} \underline{k}'_\lambda \to k \to \varinjlim_{\hat{\lambda}} k(\lambda) \to 0$$

and isomorphisms:

$$\varinjlim_{\hat{\lambda}}{}_{(n)} k(\lambda) \simeq H_{n-1}(E(\lambda);k) \qquad n \geq 2.$$

Notice that $\varinjlim_{\hat{\lambda}} k(\lambda) = 0$ unless λ is minimal in Λ_+, in which case $\varinjlim_{\lambda} k(\lambda) \simeq k$, and $\varinjlim_{\lambda}{}_{(1)} k(\lambda) = 0$

If λ is not minimal, then

$$\varinjlim_{\lambda}{}_{(1)} k(\lambda) \simeq H_o(E(\lambda);k)$$

where $\tilde{H}_\bullet$ is the augmented homology.

Together we have proved the following

<u>Proposition</u> $$\operatorname{Tor}_n^{\Lambda}(k,k) \simeq \begin{cases} k & n = 0 \\ k^{\rho} & n = 1 \\ \coprod_{\lambda\in\Lambda_+} H_{n-2}(E(\lambda);k) & \text{for } n \geq 2 \end{cases}$$

where ρ is the number of minimal elements of Λ_+.

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